Kuwait University	Math 101	Date: January 17, 2010
Dept. of Math. & Comp. Sci.	Final Exam	Duration: 120 minutes

Calculators, mobile phones, pagers and all other mobile communication equipments are not allowed.

Answer the following questions:

1. (2+2 pts.) Evaluate the following limits (if they exist).

   (a) $\lim_{x\to 0}\frac{(3x-\tan x)^2}{x^2}$ (b) $\lim_{x\to -\infty}\frac{\sqrt{1+x^2}}{2x+1}$

2. (4 pts.) Find the value of the constant A such that the following function is continuous at $x=0$.

$$f(x)=\begin{cases}3\cos x-A & \text{if } x<0,\\ 2-x^2 & \text{if } x\geq 0.\end{cases}$$

3. (4 pts.) Find $\frac{dy}{dx}$ where $y=\frac{(x^2+1)^3\tan x}{x}$. (Do NOT simplify the answer.)

4. (4 pts.) Determine a such that the average value of the function $f(x)=(ax-1)(x-a)$ on the interval $[-1,1]$ is equal to 4.

5. (4 pts.) Sketch a graph of $f(x)$ that satisfies ALL of the following conditions.

$f(0)=0$	
$\lim_{x\to-\infty} f(x)=2$	$\lim_{x\to+\infty} f(x)=+\infty$
$\lim_{x\to1^-} f(x)=+\infty$	$\lim_{x\to1^+} f(x)=-\infty$
$f'(x)<0$ on $(-\infty,0)$	$f'(x)>0$ on $(0,1)$ and $(1,+\infty)$
$f''(x)<0$ on $(-\infty,0)$ and $(1,+\infty)$	$f''(x)>0$ on $(0,1)$

6. (4 pts.) Let f be a function such that $f(0)=0$ and $f'(c)<2$, $\forall c\in\mathbb{R}$. Show that $f(x)<2x$, $\forall x>0$.

7. (4 pts.) Let $f(x)=\int_1^{2x-x^2}\frac{1}{t^4+2}\,dt$. Find the local maximum of $f(x)$.

8. (2+2 pts.) Evaluate the following integrals.

   (a) $\int_0^{\frac{\pi}{2}}\cos(\sin x)\cos x\,dx$ (b) $\int\frac{(7+\frac{1}{x^2})^9}{x^3}\,dx$

9. (4 pts.) Set up an integral for the area between the curves $y=x^2+2x$ and $y=28-x$.

10. (2+2 pts.) Set up an integral for the volume of the solid generated by revolving the region bounded by $y=x^2+4$, $y=1$, $x=0$ and $x=2$ about:

    (a) the line $y=-2$,

    (b) the line $x=4$.

1. (4 pts.) (a) $\lim_{x\to 0}\frac{(3x-\tan x)^2}{x^2} = \lim_{x\to 0}\left(\frac{3x-\tan x}{x}\right)^2 = \lim_{x\to 0}\left(\frac{3x}{x}-\frac{\tan x}{x}\right)^2 = 4.$

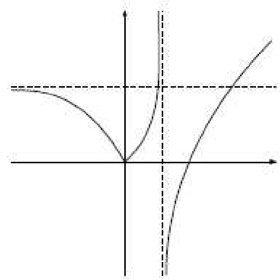
(b) $\lim_{x\to -\infty}\frac{\sqrt{1+x^2}}{2x+1} = \lim_{x\to -\infty}\frac{\sqrt{x^2(\frac{1}{x^2}+1)}}{x(2+\frac{1}{x})} = \lim_{x\to -\infty}\frac{-x\sqrt{\frac{1}{x^2}+1}}{x(2+\frac{1}{x})} = -\frac{1}{2}.$

2. (4 pts.) f is continuous at 0 if $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) = 2.$
We have $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(2-x^2) = 2$, and $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(3\cos x - A) = 3 - A.$
Thus, f is continuous at 0 when $A = 1$.

3. (4 pts.) $y' = \frac{(3(x^2+1)^2(2x)\tan x + (x^2+1)^3\sec^2 x)x - (1)(x^2+1)^3\tan x}{x^2}.$

4. (4 pts.) $f_{ave} = \frac{1}{2}\int_{-1}^{1}(ax^2-(a^2+1)x+a)\,dx = \frac{1}{2}\left(\frac{a}{3}x^3 - \frac{a^2+1}{2}x^2 + ax\right)\Big]_{-1}^{1} = \frac{4a}{3}.$
Since $f_{ave} = 4$, we have $a = 3$.

5. (4 pts.) A possible graph of $f(x)$



6. (4 pts.) $f(t)$ is differentiable $\forall\ t \in \mathbb{R}$, hence, $f(t)$ is continuous $\forall\ t \in \mathbb{R}$. Applying the MVT on $[0,x]$, $\exists\ c \in (0,x)$ such that $f'(c) = \frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x}$. Since $f'(t) < 2\ \forall\ t \in \mathbb{R}$, we derive that $\frac{f(x)}{x} < 2\ \forall\ x > 0$. Hence, $f(x) < 2x$, $\forall x > 0$.

7. (4 pts.) $f(x) = \int_1^{2x-x^2}\frac{1}{t^4+2}\,dt$. From $f'(x) = \frac{1}{(2x-x^2)^4+2}(2-2x)$ we get one critical number $x = 1$. Since $f'(x) > 0$ on $(-\infty, 1)$ and $f'(x) < 0$ on $(1, +\infty)$: at $x = 1$, f has a local maximum and $f(1) = 0$.

8. (4 pts.) Evaluate

   (a) By substitution, let $u = \sin x$, $du = \cos x\ dx$.
   $\int_0^{\frac{\pi}{2}}\cos(\sin x)\cos x\ dx = \int_0^1 \cos(u)\ du = \sin u]_0^1 = \sin 1.$

   (b) By substitution, let $u = 7 + x^{-2}$, $du = -2x^{-3}\ dx$.
   $\int\frac{(7+\frac{1}{x^2})^9}{x^3}\,dx = -\frac{1}{2}\int\frac{-2(7+\frac{1}{x^2})^9}{x^3}\,dx = -\frac{1}{2}\int u^9\ du = -\frac{1}{2}\frac{u^{10}}{10} + C = -\frac{1}{20}\left(7+\frac{1}{x^2}\right)^{10} + C.$

9. (4 pts.) To find the Points of intersection between the two curves we put $x^2 + 2x = 28 - x$. That is, $x^2 + 3x - 28 = (x-4)(x+7) = 0$. The two curves intersect at $x = -7$, $x = 4$. Therefore, the area of the region enclosed between the two curves is:

$$A = \int_{-7}^{4}((28-x)-(x^2+2x))\,dx.$$

10. (4 pts.)

   (a) About the line $y = -2$: $V = \int_0^2 \pi((2+x^2+4)^2 - (1+2)^2)\ dx = \int_0^2 \pi((x^2+6)^2 - (3)^2)\ dx$.

   (b) About the line $x = 4$: $V = \int_0^2 2\pi((4-x)((x^2+4)-(1))\ dx.$